

Multi-component generalizations of the CH equation: Geometrical Aspects, Peakons and Numerical Examples

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Abstract

The Lax pair formulation of the two-component Camassa-Holm equation (CH2) is generalized to produce an integrable multi-component family, CH(n,k), of equations with n components and $1 \leq |k| \leq n$ velocities. All of the members of the CH(n,k) family show fluid-dynamics properties with coherent solitons following particle characteristics. We determine their Lie-Poisson Hamiltonian structures and give numerical examples of their soliton solution behaviour. We concentrate on the CH(2, k) family with one or two velocities, including the CH(2,-1) equation in the Dym position of the CH2 hierarchy. A brief discussion of the CH(3,1) system reveals the underlying graded Lie-algebraic structure of the Hamiltonian formulation for CH(n,k) when $n \geq 3$.

1 Introduction

Purpose.

This paper develops the Lax formulation for an integrable family of equations that contains the multi-component generalizations CH(n,k) of the CH equation with n momentum components convected by $1 \leq |k| \leq n$ velocities. In the CH(n,k) family, CH is designated by CH(1,1) and CH2 by CH(2,1). We also consider the Lie-Poisson Hamiltonian properties of several other members of the CH(n,k) family, particularly CH(2,1), CH(2,2), CH(2,-1), CH(3,1) and CH(3,2). The CH(2,-1) system may be regarded as a two-component generalization for CH of the Dym equation in the CH hierarchy. All the CH(n,k) equations for a given value of n share the same spectral problem and therefore belong to the same integrable hierarchy. As might be expected, higher-order powers of the spectral parameter appear in the Lax formulation of CH(n,k) for greater values of n . When compared to rigid rotations, the CH(2,1) shallow water system recovers the heavy top equations and the CH(2,2) system recovers the equations for a rigid body in a potential field. The CH(3,1) system reveals the underlying graded Lie-algebraic structure of the Hamiltonian formulation for CH(n,k). Examples of numerical solutions of these integrable systems of equations illustrate their interesting dynamical properties, in which soliton trains emerge from spatially confined initial conditions and interact with each other in a variety of different ways for the various CH(n,k) systems investigated here. Many open problems arise and we attempt to sketch some of the opportunities for future research in the conclusion section.

1.1 Brief review of the Camassa-Holm (CH) equation

The CH equation [6, 7]

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad (1.1)$$

governs the evolution of the function $u(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, interpreted as a shallow water fluid velocity. When the linear dispersion parameter $\omega \in \mathbb{R}$ vanishes ($\omega = 0$) the CH equation (1.1) admits *peakon* solutions. Peakons are

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nonanalytic solitons that superpose as

$$u(x, t) = \frac{1}{2} \sum_{a=1}^N p_a(t) e^{-|x - q_a(t)|}, \quad (1.2)$$

for sets $\{p(t)\}$ and $\{q(t)\}$ satisfying a system of completely integrable canonical Hamiltonian equations.

For $\omega \neq 0$, the CH equation (1.1) describes the unidirectional propagation of dispersive shallow water waves over a flat bottom at one order higher than KdV in the standard asymptotic expansion of the Euler fluid equations with a free surface in a certain Galilean frame (mean wave speed) [6, 7, 16, 17, 39, 40]. It also describes axially symmetric waves in a hyperelastic rod [15]. The inverse scattering problem for CH is treated, e.g., in [4, 11, 12, 13]. For a brief history of the developments and results about the CH equation of relevance in the present paper, one may consult the recent review [26] and references therein.

The scope of the variety of mathematical interpretations of CH may be gleaned by rewriting it in various equivalent forms, many of which have been discovered several times before and each of which can be a point of departure for further investigation. For example, CH may be treated variously as: a fluid motion equation; a mathematical model of shallow water wave breaking; a vanishing Lie derivative, describing invariance of a 1-form density under the flow of a vector field related to it by inversion of the Helmholtz operator; a nonlocal characteristic equation; an Euler-Poincaré equation describing geodesic motion on the diffeomorphism group with respect to the metric defined by the H^1 norm on the tangent space of vector fields; a Lie-Poisson Hamiltonian system describing coadjoint motion on the Bott-Virasoro Lie group; a bi-Hamiltonian system; a compatibility equation for a linear system of two equations in a Lax pair, etc.

Our own point of departure and primary emphasis in this paper is in treating CH as a member of a family of integrable evolutionary equations associated with a certain class of energy-dependent isospectral eigenvalue problems of polynomial order. We shall also discuss the geometrical interpretations and varieties of solution behaviour of several members of the hierarchy. To establish notation, we begin by writing CH in several of its various forms and discussing its properties that are relevant here. Most of these properties were already established in [6, 7].

CH as a fluid motion equation. CH may be written in the form of a *fluid motion equation*, as

$$u_t + uu_x = -P_x, \quad (1.3)$$

with *pressure* P given by the convolution

$$P = K * (u^2 + \frac{1}{2}u_x^2 + \omega u) \quad \text{with kernel} \quad K(x, y) = \frac{1}{2} \exp(-|x - y|). \quad (1.4)$$

The kernel K is the Green's function for the 1D Helmholtz operator, $(1 - \partial_x^2)$, and is also the shape of the peakon profile in equation (1.2).

Lie derivative, or characteristic form of CH. Upon introducing a *momentum* variable

$$m = u - u_{xx} = (1 - \partial_x^2)u, \quad (1.5)$$

CH expresses the *vanishing Lie derivative* condition for invariance of its momentum, as a 1-form density, along characteristics of an associated velocity vector field,

$$(\partial_t + \mathcal{L}_u)((m + \omega) dx^2) = (m_t + um_x + 2(m + \omega)u_x) dx^2 = 0, \quad \text{with} \quad u(x, t) = K * m. \quad (1.6)$$

This form of CH may be interpreted as the condition for the 1-form density $(m + \omega) dx^2$ to be preserved (frozen in) under the flow of a characteristic velocity $dx/dt = u(x(t), t)$, namely,

$$\frac{d}{dt}((m + \omega) dx^2) = 0, \quad \text{along} \quad \frac{dx}{dt} = u(x(t), t) = K * m. \quad (1.7)$$

In the parlance of fluids, equation (1.6) is the **Eulerian** form of the invariance, while equation (1.7) is its equivalent **Lagrangian** form. As with the evolution of vorticity according to Euler's equations for incompressible fluid motion, the relation between the velocity of the flow and the property it carries is *nonlocal*. For the Euler fluid equations, the velocity carries the vorticity, related to the velocity by the convolution in the Biot-Savart law, which inverts the curl operation. For CH, the momentum is related to the velocity by convolution with the kernel K in (1.4), which inverts the Helmholtz operator. When expressed in terms of the momentum, the peakon velocity solution (1.2) of dispersionless CH for $\omega = 0$ becomes a sum over delta functions, supported on a set of points moving on the real line. That is, the peakon velocity solution (1.2) implies

$$m(x, t) = \sum_{a=1}^N p_a(t) \delta(x - q_a(t)), \quad (1.8)$$

because of the relation $(1 - \partial_x^2)(\frac{1}{2}e^{-|x-y|}) = \delta(x - y)$ for the kernel K in (1.4). As shown in [27], the CH peakon solution (1.8) is geometrically the cotangent-lift momentum map for the left action of the diffeomorphisms $\text{Diff}(\mathbb{R})$ on a set of N points on the real line.

The Euler-Poincaré and Lie-Poisson properties of CH The geometric properties of CH in (1.1) follow from Hamilton's principle with a Lagrangian $l(u) : \mathfrak{X}(\mathbb{R}) \rightarrow \mathbb{R}$ by using the Euler-Poincaré theory [28]. Its associated Lie-Poisson Hamiltonian formulation in terms of $m \in \mathfrak{X}^*(\mathbb{R})$ then emerges from a Legendre transformation. Here, the velocity vector field $u \in \mathfrak{X}(\mathbb{R}) \simeq T\text{Diff}(\mathbb{R})/\text{Diff}(\mathbb{R})$ in the Lagrangian $l(u)$ is right-invariant under $\text{Diff}(\mathbb{R})$, the diffeomorphisms of the real line. The Lagrangian for CH is $l(u) = \frac{1}{2}\|u\|_{H^1}^2$, the H^1 norm on these vector fields; which provides the metric for the interpretation of CH solutions as geodesic motion on $\text{Diff}(\mathbb{R})$. The variational derivative $\delta l/\delta u \in \mathfrak{X}^*(\mathbb{R})$, in the space of real-valued 1-form densities dual to $\mathfrak{X}(\mathbb{R})$ yields the CH momentum,

$$m = \frac{\delta l}{\delta u} = u - u_{xx}.$$

The Lie-Poisson Hamiltonian formulation of CH follows by Legendre transforming the Euler-Poincaré equation for right-invariant vector fields, as

$$\frac{d}{dt} \frac{\delta l}{\delta u} = -\text{ad}_u^* \frac{\delta l}{\delta u} \implies m_t = -(\partial_x m + m \partial_x) \frac{\delta h}{\delta m} \quad (1.9)$$

with

$$h(m) = \langle m, u \rangle - l(u), \quad \frac{\delta h}{\delta m} = u, \quad \frac{\delta h}{\delta u} = 0 = m - \frac{\delta l}{\delta u},$$

where $\langle \cdot, \cdot \rangle : \mathfrak{X}^*(\mathbb{R}) \times \mathfrak{X}(\mathbb{R}) \rightarrow \mathbb{R}$ denotes L^2 pairing on the real line. For more detail of the Lie-Poisson structure of CH, see [28]. For its interpretation as coadjoint motion on the Bott-Virasoro Lie group, see [49]. The Euler-Poincaré and Lie-Poisson structure of the CH equation in (1.9) makes it clear how to generalize it to higher dimensions. The result is the **EPDiff equation** in n dimensions, given for $\omega = 0$ by

$$\partial_t m_i = -(\partial_j m_i + m_j \partial_i) u^j \quad \text{with velocity components} \quad \frac{\delta h}{\delta m_j} = u^j, \quad (1.10)$$

where $i, j = 1, 2, \dots, n$.

The bi-Hamiltonian property of CH. The one-dimensional CH equation (1.1) may be written in bi-Hamiltonian form as

$$m_t = -(\partial_x - \partial_x^3) \frac{\delta H_2[m]}{\delta m} = -\left(\partial_x(m + \omega) + (m + \omega) \partial_x\right) \frac{\delta H_1[m]}{\delta m}, \quad (1.11)$$

where the two Hamiltonians are given by

$$H_1[m] = \frac{1}{2} \int m u \, dx \quad \text{and} \quad H_2[m] = \frac{1}{2} \int (u^3 + u u_x^2 + 2\omega u^2) \, dx. \quad (1.12)$$

The integration is over the real line, for functions that decay sufficiently rapidly as $|x| \rightarrow \infty$, and over one period, for periodic functions. By Magri's theorem [45], the bi-Hamiltonian property of CH implies an infinite sequence of conservation laws, obtained by a recursion relation.

The Lax pair for CH. Its bi-Hamiltonian property also implies that the CH equation (1.1) admits a **Lax pair representation**, given by [6, 7]

$$\Psi_{xx} = \left(\frac{1}{4} + \lambda(m + \omega) \right) \Psi, \quad (1.13)$$

$$\Psi_t = \left(\frac{1}{2\lambda} - u \right) \Psi_x + \frac{u_x}{2} \Psi + \gamma \Psi, \quad (1.14)$$

where ω, γ are arbitrary real constants and the eigenvalue λ is independent of time. The compatibility of the Lax pair for constant λ means that the eigenvalue equation in (1.13) is **isospectral**. That is, its spectrum is invariant under the flow of the CH equation.

Many papers have been written to explore various features of the original single-component CH equation. A brief history of its exploration is recounted, for example, in [26]. See also [25] for a recent discussion of its singular peakon solutions.

Plan of the paper

Section 2 continues the introduction of our subject by discussing the extension of CH to the integrable CH(n,k) systems consisting of n components (momentum densities) and $1 \leq |k| \leq n$ velocities that result from the isospectral problem in (2.1) and (2.2). The CH(n,k) hierarchy may be written in a compact universal form that aids in the physical interpretation of its various equations as continuum flows and is reminiscent of the Virasoro structure of the one-component CH equation found in [49]. Section 3 reviews the properties of the two-component CH(2,1) system in (3.1)-(3.2) and then discusses the CH(2,2) system, both of which are fluid systems. Section 4 provides two other examples of equations in the CH2 hierarchy: (i) the CH(2, -1) system, in the position of the CH hierarchy corresponding to the Dym equation in the CH hierarchy; and (ii) the CH(2,1) system with two time variables. Section 5 closes the paper by giving a brief summary of its main points and indicating some open problems for future research.

2 A hierarchy of multi-component integrable extensions of CH

The main feature of the Inverse Scattering Transform (IST) for the two-component generalization of the CH equation (CH2) is that its spectral problem is Schrödinger's equation with an 'energy dependent' potential, in which higher-order powers of the spectral parameter appear. For the history and development of the IST method with energy dependent potentials, one may consult [41, 38, 53, 2, 47] and the references therein.

A hierarchy of multi-component generalizations of CH may be obtained by considering an extension of the Lax pair (1.13), (1.14) that preserves its form, but replaces its coefficients by polynomials in the scattering parameter λ , as in [35],

$$\Psi_{xx} = Q(x, \lambda) \Psi, \quad (2.1)$$

$$\Psi_t = -U(x, \lambda) \Psi_x + \frac{1}{2} U_x(x, \lambda) \Psi, \quad (2.2)$$

where the potential $Q(x, \lambda)$ has the following energy dependence

$$Q(x, \lambda) = \lambda^n q_n(x) + \lambda^{n-1} q_{n-1}(x) + \dots + \lambda q_1(x) + \frac{1}{4}, \quad (2.3)$$

$$U(x, \lambda) = u_0(x) + \frac{u_1(x)}{\lambda} + \dots + \frac{u_k(x)}{\lambda^k}. \quad (2.4)$$

The compatibility condition for (2.1), (2.2) gives the following equation,

$$Q_t + (\partial_x Q + Q \partial_x)U = \frac{1}{2}U_{xxx}, \quad (2.5)$$

whose form is reminiscent of the Virasoro structure of the one-component CH equation found in [49].

Following this important clue to the nature of the equations in this hierarchy, we rewrite the equation geometrically as the Lie derivative \mathcal{L}_U of the 1-form density $(Q dx^2)$ with respect to the vector field U as

$$(\partial_t + \mathcal{L}_U)(Q dx^2) = \frac{1}{2}(dU_{xx})dx, \quad (2.6)$$

whose relation to continuum flows may be emphasized by rewriting it in characteristic form as

$$\frac{d}{dt} \left(Q(x(t), t) dx(t)^2 \right) = \frac{1}{2}(dU_{xx})dx(t), \quad \text{along} \quad \frac{dx}{dt} = U(x(t), t). \quad (2.7)$$

Interpretation. The left hand side of equations (2.6) and (2.7) represents sweeping of the wave momentum density Q by the flow of the velocity vector field U , while the right hand side represents dispersion of the wave and forces that cause motion relative to the flow lines of U .

CH(n,k) family of equations. Upon substituting the expansions in λ and λ^{-1} from (2.3) and (2.4), respectively, into equation (2.5), one obtains a chain of n evolution equations with $k+1$ differential relations for the $n+k+1$ variables $q_1, q_2, \dots, q_n, u_0, u_1, \dots, u_k$ (n and k are arbitrary positive, or negative, integers):

$$\begin{aligned} q_{n-r,t} &= - \sum_{s=\max(0, r-k)}^r (\partial_x q_{n-s} + q_{n-s} \partial_x) u_{r-s}, \quad r = 0, 1, \dots, n-1, \\ 0 &= \frac{1}{2}(\partial_x - \partial_x^3)u_r + \sum_{s=1}^{\min(n, k-r)} (\partial_x q_s + q_s \partial_x) u_{r+s}, \\ &\quad r = 0, 1, \dots, k-1, \\ 0 &= (\partial_x - \partial_x^3)u_k. \end{aligned} \quad (2.8)$$

The differential relations in the middle equation of (2.8) contain the same (Hamiltonian) operators as in the bi-Hamiltonian structure (1.11).

System (2.8) is similar to the hydrodynamic chain studied in a series of papers [54, 47, 48], and to other CH generalizations [21, 51, 19, 42, 10]. In the present paper, we shall examine the geometric structure and numerical solution behavior of several examples of equations from the CH(n,k) family of equations resulting from the isospectral problem (2.1) and (2.2).

2.1 CH(n,1) system

The simplest chain system with n coupled equations arises from the general framework (2.8) for $k=1$. This system is denoted CH(n,1). In this case, the Lax pair (2.1) - (2.2) becomes

$$Q(x, \lambda) = -\lambda^n \rho^2 + \lambda^{n-1} q_{n-1} + \dots + \lambda q_1 + \frac{1}{4}, \quad (2.9)$$

$$U(x, \lambda) = -\frac{1}{2\lambda} + u. \quad (2.10)$$

There is one differential relation, $q_1 = u - u_{xx} + \text{const}$. Compatibility of the corresponding Lax pair in this case results in a chain of equations in the following form for $p = 1, 2, \dots, n$,

$$\partial_t q_p + (\partial_x q_p + q_p \partial_x)u - \frac{1}{2} \partial_x q_{p+1} = 0 \quad \text{with} \quad q_n = -\rho^2 \quad \text{and} \quad q_{n+1} = 0. \quad (2.11)$$

2.2 CH(n,2) system

The Lax pair (2.1) - (2.2) with

$$Q = \lambda^n q_n + \lambda^{n-1} q_{n-1} + \dots + \lambda q_1 + \frac{1}{4}, \quad (2.12)$$

$$U = -\frac{1}{2\lambda^2} + \frac{u_1}{\lambda} + u_0, \quad (2.13)$$

generates the coupled n -component CH(n,2) system for $p = 1, 2, \dots, n$,

$$\partial_t q_p + (\partial_x q_p + q_p \partial_x) u_0 + (\partial_x q_p + q_p \partial_x) u_1 - \frac{1}{2} q_{p+2,x} = 0, \quad (2.14)$$

with $q_{n+1} = q_{n+2} = 0$. The differential relations in (2.8) give q_1 and q_2 in terms of u_0 and u_1 in (3.5) and (3.6), below. The others (q_3, \dots, q_n) are independent.

2.3 CH(n,k) system

The pattern continues for CH(n,k) upon including more velocities, u_0, u_1, \dots, u_{k-1} , with $u_k = -\frac{1}{2}$. The p -th equation in the integrable n -component system CH(n,k) is obtained from the Lax pair (2.1) - (2.2) as

$$\partial_t q_p + \sum_{j=0}^{k-1} (\partial_x q_{p+j} + q_{p+j} \partial_x) u_j - \frac{1}{2} q_{p+k,x} = 0, \quad (2.15)$$

$$\text{with } q_{n+1} = q_{n+2} = \dots = q_{n+k} = 0.$$

The solutions for u_0, \dots, u_{k-1} are updated at each time step from q_1, \dots, q_k by imposing the $k+1$ differential relations in (2.8).

In the remainder of the paper we will concentrate most of our attention on the two-component systems, including CH(2,1), referred to simply as CH2, as well as CH(2,2) and CH(2,-1). (In the CH(n,k) notation, *negative* values of k refer to positive powers of λ in the expansion of U in (2.13).) However, in the case $n = 3$ the examples CH(3,1) and CH(3,2) will reveal the graded structure of the Hamiltonian formulations of all of the equations in the CH(n,k) family. We will also show a few numerical solutions of these equations that provide crucial insight into their pulse-like evolutionary behavior, inviting further investigation.

3 Examples of two-component CH systems

3.1 Example: CH2, or CH(2,1) for $n = 2, k = 1$

We denote $u_0 \equiv u$, $q_1 \equiv q$ and $q_2 \equiv \pm \rho^2$, and choose $u_1 = -1/2$. In this notation, the CH(2,1) system can be written in the form

$$q_t + (\partial_x q + q \partial_x) u \mp \rho \rho_x = 0, \quad (3.1)$$

$$\rho_t + (u \rho)_x = 0, \quad (3.2)$$

where $q = u - u_{xx} + \omega$ and ω is an arbitrary constant.

This is generally known as the CH2 system, and it has been studied extensively. The last term in the motion equation in CH2 has a choice of sign (\mp). For the positive choice, the CH2 equation may be regarded as a model of shallow water waves [14, 29]. Its generalization to higher dimensions is immediate.

Spectral problem for CH2. The *spectral problem for CH2* is, as in equations (2.9) - (2.10),

$$\Psi_{xx} = \left(-\lambda^2 \rho^2(x) + \lambda q(x) + \frac{1}{4} \right) \Psi. \quad (3.3)$$

This spectral problem is a type of Schrödinger equation with an ‘energy dependent’ potential, i.e. it is quadratic in terms of the spectral parameter and moreover the potential functions multiply the spectral parameter (the so-called weighted problem). There are some common features with Sturm-Liouville spectral problems, see for example [38, 41, 53]. An ‘energy dependent’ spectral problem also appears in the inverse scattering transform of an integrable generalization of the Bousinesq equation (Kaup-Bousinesq equation) [41].

Brief history of the CH2 equation. The system (3.1)-(3.2) representing a two-component generalization of the CH equation was initially introduced in [50] as a tri-Hamiltonian system. It was studied further by others, see, e.g., [44, 8, 20, 14, 29, 22]. The known applications of the CH2 model are the following.

- In the context of shallow water theory, u can be interpreted as the horizontal fluid velocity and ρ is related to the water elevation in the first approximation [14, 36].
- In Vlasov plasma models, CH2 describes the closure of the kinetic moments of the single-particle probability distribution for geodesic motion on the symplectomorphisms [55, 31, 32].
- In the large-deformation diffeomorphic approach to image matching, the CH2 equation is summoned in a type of matching procedure called metamorphosis [33].

The same CH2 system appears as a member of the hierarchy of hydrodynamic chains studied in [54]. Its analytical properties such as well-posedness and wave breaking were studied in [18, 34, 56, 22, 9] and others.

Geometrically, the original CH equation may be interpreted as governing geodesic motion for an H^1 metric that is invariant under the Virasoro group, as found in [49]. Because of this property, CH is geometrically reminiscent of the Euler rigid body equations, which describe geodesic motion on the rotation group with respect to the metric supplied by the moment of inertia. The geometric interpretation of CH2 is similar: CH2 is the equation for geodesic motion on the semidirect-product Lie group of diffeomorphisms acting on densities, with respect to the H^1 on the horizontal velocity and the L^2 norm on the elevation. This is analogous to the finite-dimensional case of an ellipsoidal underwater vehicle (UWV), whose motion may be modelled as geodesics on the Euclidean group of rigid body rotations and translations. This finite-dimensional model of the UWV has the same Lie-Poisson bracket as for the Hamiltonian description of the heavy top, so CH2 may also be interpreted analogously to the heavy-top equations. For additional discussions of geometric aspects of the CH2 system we refer to [30, 43, 33].

In general, one can show that small initial data of the CH2 system develop into global solutions, while for some initial data wave breaking occurs [18, 14, 34, 56, 22, 9]. It is interesting that only the plus sign (+) in (3.1) corresponds to a positively defined Hamiltonian and straightforward physical applications to shallow water waves. It would be interesting to know the physical interpretation of the model with the choice of the minus sign in (3.1), since this case is also integrable.

Solutions of CH2 for dam-break initial conditions. Figure 3.1 plots the evolution of CH2 solutions for (u, ρ) governed by equations (3.1-3.2) with the + sign choice in the periodic domain $[-L, L]$ with *dam-break initial conditions* given by

$$u(x, 0) = 0, \quad \rho(x, 0) = 1 + \tanh(x + a) - \tanh(x - a), \quad (3.4)$$

where $a \ll L$.

The dam-break involves a body of water of uniform depth, retained behind a barrier, in this case at $x = \pm a$. If this barrier is suddenly removed at $t = 0$, then the water would flow downward and outward under gravity. The problem is to find the subsequent flow and determine the shape of the free surface. This question is addressed in

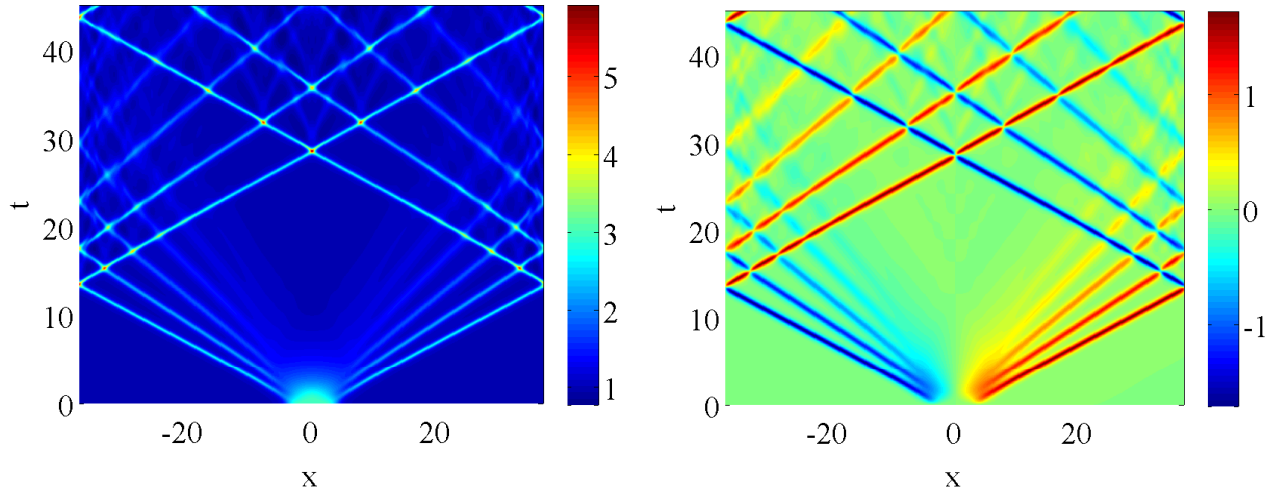


Figure 3.1: Dam-break results for the CH(2,1) system in equations (3.1-3.2) show evolution of the density ρ (left panel) and velocity u (right panel), arising from initial conditions (3.4) in a periodic domain. The color bars show positive density on the left and both positive and negative velocity on the right. The soliton solutions are seen to emerge symmetrically leftward and rightward after a finite time, and the evolution of both variables generates more and more solitons propagating in both directions as time progresses. Figures are courtesy of L. Ó Náráigh.

the context of shallow-water theory, e.g., by Acheson [1], and thus serves as a typical hydrodynamic problem of relevance for CH2 solutions with the $+$ sign choice in (3.1).

The CH2 system invites further generalizations and applications. For example, its two-time generalization is presented in §4.2 of this paper.

3.2 Example: CH(2,2) for $n = k = 2$

CH(2,2) designates the case of two momentum densities q_1 and q_2 and two velocities u_0 and u_1 . The choice $u_2 = -1/2$ automatically solves one of the relations in (2.8). The other two differential relations in (2.8) can then be integrated spatially in x , to find the relationships between the momenta and the velocities, as

$$q_1 = u_1 - u_{1,xx} + \omega_1, \quad (3.5)$$

$$q_2 = u_0 - u_{0,xx} + 3u_1^2 - u_{1,x}^2 - 2u_1u_{1,xx} + 4\omega_1u_1 + \omega_2, \quad (3.6)$$

where ω_1 and ω_2 are constants of integration that depend on boundary conditions. For CH(2,2) the evolutionary system (2.8) yields equations for q_1 and q_2 given by

$$q_{1,t} + (\partial_x q_1 + q_1 \partial_x) u_0 + (\partial_x q_2 + q_2 \partial_x) u_1 = 0, \quad (3.7)$$

$$q_{2,t} + (\partial_x q_2 + q_2 \partial_x) u_0 = 0. \quad (3.8)$$

These equations may be solved by first updating q_2 , then q_1 in (3.7) and (3.8), followed by inverting the Helmholtz operator twice, first for u_1 in (3.5) and then for u_0 in (3.6).

Dam-break equivalent problem in q_2 for CH(2,2). Figure 3.2 shows the evolution of pulses in the velocity variables u_0 (left panel) and u_1 (right panel). These pulses arise from an initially localized disturbance in q_2 with a tanh-squared profile, for $\omega_1 = \omega_2 = 0$,

$$q_2(x, 0) = [1 + \tanh(x + 1) - \tanh(x - 1)]^2, \\ u_1(x, 0) = q_1(x, 0) = 0.01,$$

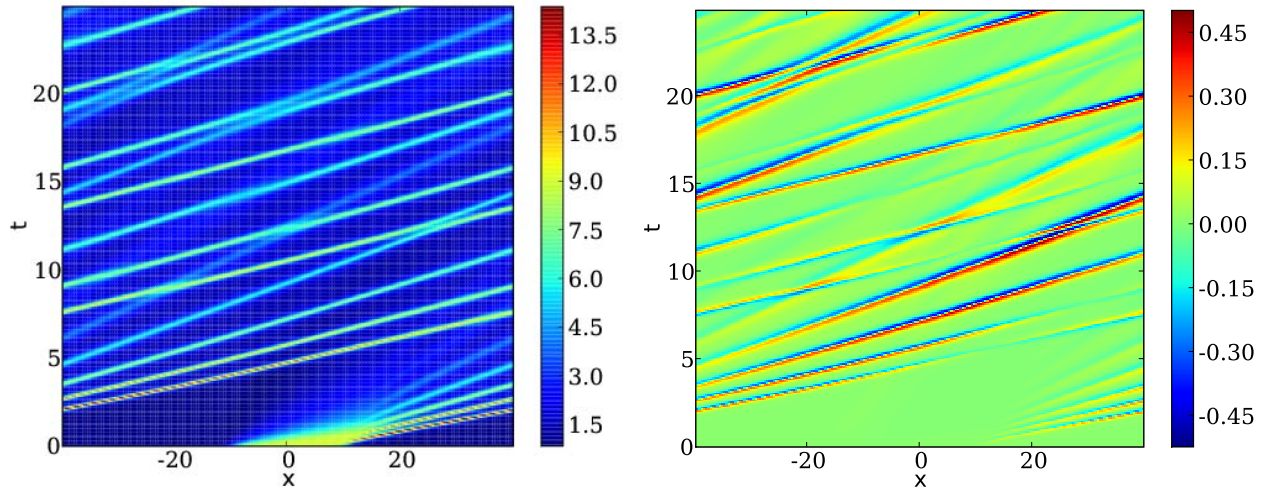


Figure 3.2: This figure shows the dam-break solution behavior of the CH(2,2) system (3.7)-(3.8), which should be compared with the corresponding results in Figure 3.1 for the system of CH(2,1) equations in (3.1)-(3.2). The dam-break results for the CH(2,2) system show evolution of the velocity variables u_0 (left panel) and u_1 (right panel), arising from initial conditions for $\sqrt{q_2}$ similar to (3.4) in a periodic domain. A “fan” of sequentially smaller, slower pulses is emitted rightward, after which the leading pulses overtake the slower lagging pulses in the periodic domain and suffer elastic collisions that show a variety of different phase shifts, apparently depending on their relative speeds. Each confined positive pulse in u_0 is accompanied by dipole (\pm) excitations in u_1 , whose polarity may occasionally reverse from (\pm , red-blue) to (\mp , blue-red) and back. Figures are courtesy of J. R. Percival.

that interacts with a constant mean flow in the velocity field u_1 . The constants ω_1, ω_2 are set to zero. The initial confined pulse $q_2(x, 0)$ corresponds to a confined pulse in u_0 that propagates steadily rightward and generates a structured dipole pulse in u_1 that accompanies the u_0 pulse, but may oscillate in polarity as it propagates through a background “fan” of smaller slower pulses. This is an interesting scenario whose dynamics will be investigated further elsewhere.

3.3 A semidirect product interpretation of the example CH(2,2)

The system (3.7-3.8) for CH(2,2) may be rewritten as

$$(\partial_t + \mathcal{L}_{u_0})(q_2 dx^2) = 0, \quad (\partial_t + \mathcal{L}_{u_0})(q_1 dx^2) + \mathcal{L}_{u_1}(q_2 dx^2) = 0, \quad (3.9)$$

where \mathcal{L}_u denotes Lie derivative with respect to the vector field u .

To understand system (3.9) better geometrically, we shall rederive it from Hamilton’s principle with a Lagrangian defined on the semidirect product Lie algebra of vector fields $l(u_0, u_1) : \mathfrak{X}_0(\mathbb{R}) \otimes \mathfrak{X}_1(\mathbb{R}) \rightarrow \mathbb{R}$. Here $u_0 \in \mathfrak{X}_0(\mathbb{R})$ and $u_1 \in \mathfrak{X}_1(\mathbb{R})$ are right-invariant smooth vector fields on the real line \mathbb{R} . Vector fields in $\mathfrak{X}_0(\mathbb{R})$ act on themselves and on $\mathfrak{X}_1(\mathbb{R})$ by vector cross product, while the action of $\mathfrak{X}_1(\mathbb{R})$ on itself is assumed to be by simple vector addition. That is, vector fields in $\mathfrak{X}_1(\mathbb{R})$ are *advected quantities*, see, e.g., [28].

Definitions: Semi-direct product. The semidirect product Lie algebra action $\mathfrak{X}_0(\mathbb{R}) \otimes \mathfrak{X}_1(\mathbb{R})$ is defined by

$$[(X_0, X_1), (Y_0, Y_1)] = ([X_0, Y_0], [X_0, Y_1] + [X_1, Y_0]), \quad (3.10)$$

where $[\cdot, \cdot]$ is the commutator of vector fields in natural notation. This commutator defines the adjoint action

$$\text{ad}_{(X_0, X_1)}(Y_0, Y_1) = -[(X_0, X_1), (Y_0, Y_1)] = (\text{ad}_{X_0} Y_0, \text{ad}_{X_0} Y_1 + \text{ad}_{X_1} Y_0) \quad (3.11)$$

and the L^2 pairing with $(\alpha, \beta) \in \mathfrak{X}_0^* \otimes \mathfrak{X}_1^*$ yields the coadjoint action

$$\begin{aligned}
 \langle \text{ad}_{(X_0, X_1)}^*(\alpha, \beta), (Y_0, Y_1) \rangle &= \langle (\alpha, \beta), \text{ad}_{(X_0, X_1)}(Y_0, Y_1) \rangle \\
 &= \langle (\alpha, \beta), (\text{ad}_{X_0} Y_0, \text{ad}_{X_0} Y_1 + \text{ad}_{X_1} Y_0) \rangle \\
 &= \langle \alpha, \text{ad}_{X_0} Y_0 \rangle + \langle \beta, \text{ad}_{X_0} Y_1 + \text{ad}_{X_1} Y_0 \rangle \\
 &= \langle \text{ad}_{X_0}^* \alpha + \text{ad}_{X_1}^* \beta, Y_0 \rangle + \langle \text{ad}_{X_0}^* \beta, Y_1 \rangle \\
 &= \langle (\text{ad}_{X_0}^* \alpha + \text{ad}_{X_1}^* \beta, \text{ad}_{X_0}^* \beta), (Y_0, Y_1) \rangle.
 \end{aligned} \tag{3.12}$$

See [5] for more background and derivations of the formulas for the adjoint and coadjoint actions of the semidirect product Lie group $\text{Diff}_0(\mathbb{R}) \otimes \text{Diff}_1(\mathbb{R})$ and its Lie algebra of right-invariant vector fields $\mathfrak{X}_0 \otimes \mathfrak{X}_1$.

We shall rederive system (3.9) from Hamilton's principle by using the Euler-Poincaré theory, as reviewed for continuum mechanics, e.g., in [28]. We then pass to its Hamiltonian formulation in terms of a Lie-Poisson bracket by performing a Legendre transformation. In the Euler-Poincaré framework, we have the following.

Theorem 1 (Euler-Poincaré formulation of CH(2,2)).

Hamilton's principle $\delta S = 0$ with $S = \int l(u_0, u_1) dt$ yields the CH(2,2) system in equations (3.9) for the Lagrangian

$$l(u_0, u_1) = \int_{\mathbb{R}} u_0 u_1 + u_{0,x} u_{1,x} + \omega_1 u_0 + \omega_2 u_1 + 2u_1 (u_1^2 + u_{1,x}^2) dx, \tag{3.13}$$

for constrained variations of u_0 and u_1 of the semidirect product form in (3.10), consisting of

$$\delta u_0 = \partial_t \xi_0 + [u_0, \xi_0] = \partial_t \xi_0 - \text{ad}_{u_0} \xi_0, \tag{3.14}$$

$$\delta u_1 = \partial_t \xi_1 + [u_0, \xi_1] + [u_1, \xi_0] = \partial_t \xi_1 - \text{ad}_{u_0} \xi_1 - \text{ad}_{u_1} \xi_0. \tag{3.15}$$

Proof. By direct calculation, Hamilton's principle with the Lagrangian (3.13) implies

$$\begin{aligned}
 0 = \delta S &= \int \langle q_1, \delta u_0 \rangle + \langle q_2, \delta u_1 \rangle dt \\
 &= \int \langle q_1, \partial_t \xi_0 - \text{ad}_{u_0} \xi_0 \rangle + \langle q_2, \partial_t \xi_1 - \text{ad}_{u_0} \xi_1 - \text{ad}_{u_1} \xi_0 \rangle dt \\
 &= - \int \langle \partial_t q_1 + \text{ad}_{u_0}^* q_1 + \text{ad}_{u_1}^* q_2, \xi_0 \rangle + \langle \partial_t q_2 + \text{ad}_{u_0}^* q_2, \xi_1 \rangle dt,
 \end{aligned}$$

where q_1 and q_2 are given in (3.5) and (3.6), respectively, and we have used the semidirect product Lie algebra action defined in (3.10). The coadjoint operation ad^* is defined using the L^2 pairing as in, e.g.,

$$\langle q_2, \text{ad}_{u_0} \xi_1 \rangle = \langle \text{ad}_{u_0}^* q_2, \xi_1 \rangle = \langle \mathcal{L}_{u_0} q_2, \xi_1 \rangle = \langle (\partial_x q_2 + q_2 \partial_x) u_0, \xi_1 \rangle, \tag{3.16}$$

which is a repeated pattern in the system (3.9).

□

Legendre transform to the CH(2,2) Hamiltonian formulation. The Legendre transform for CH(2,2) is given by

$$h(q_1, q_2) = \langle (q_1, q_2), (u_0, u_1) \rangle - l(u_1, u_2), \quad \frac{\delta h}{\delta m} = u, \quad \frac{\delta h}{\delta u} = 0 = m - \frac{\delta l}{\delta u}, \tag{3.17}$$

where $\langle \cdot, \cdot \rangle$ denotes L^2 pairing on the real line. The corresponding variations yield

$$\delta h = \langle (u_0, u_1), (\delta q_1, \delta q_2) \rangle + \left\langle \left(q_1 - \frac{\delta l}{\delta u_0}, q_2 - \frac{\delta l}{\delta u_1} \right), (\delta u_0, \delta u_1) \right\rangle. \tag{3.18}$$

Hence the pairs (q_1, q_2) and (u_0, u_1) are dual variables with respect to the Legendre transform of the CH(2,2) Lagrangian in (3.13). Thus, the differential relations in the Lax pair formulation of CH(2,2) appearing in (3.7) and (3.8) Legendre transform into dual momenta for the Hamiltonian formulation.

Lie-Poisson Hamiltonian form of CH(2,2). The CH(2,2) system in (3.7-3.8) or (3.9) may now be cast into Lie-Poisson Hamiltonian form, as

$$\begin{bmatrix} \partial_t q_1 \\ \partial_t q_2 \end{bmatrix} = - \begin{bmatrix} \partial_x q_1 + q_1 \partial_x & \partial_x q_2 + q_2 \partial_x \\ \partial_x q_2 + q_2 \partial_x & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta q_1 = u_0 \\ \delta h / \delta q_2 = u_1 \end{bmatrix}, \quad (3.19)$$

The Hamiltonian operator yields the **Lie-Poisson bracket** defined on the dual to the **semidirect product** Lie algebra of vector fields $\mathfrak{X}_0(\mathbb{R}) \ltimes \mathfrak{X}_1(\mathbb{R})$. The Lie algebra action for the semidirect product is defined by (3.10) and dual coordinates are $q_1 \in \mathfrak{X}_0^*(\mathbb{R})$ and $q_2 \in \mathfrak{X}_1^*(\mathbb{R})$.

Remark 1. The Lie-Poisson Hamiltonian form (3.19)

$$\partial_t(q_2 dx^2) = -\mathcal{L}_{u_0}(q_2 dx^2)$$

may be interpreted as saying that the 1-form density $(q_2 dx^2)$ evolves in time t by the action of $\text{Diff}_0(\mathbb{R})$ on its initial conditions. That is,

$$\frac{d}{dt}(q_2 dx^2) = 0 \quad \text{along} \quad \frac{dx}{dt} = u_0(t, x(t)).$$

Quantities that evolve this way to remain invariant along the characteristic paths of a flow velocity in ideal fluid mechanics are said to be **advected**, or **frozen into the flow**. Equation (3.9) shows that the process is not passive, though; because the dynamics of q_1 is affected by a force depending on q_2 and the corresponding velocities that are obtained from the differential relations in (3.7) and (3.8).

The other Hamiltonian structure for this example would be interesting to know. However, knowing it is not necessary for the sake of generating its integrable hierarchy, because we already have its isospectral problem and Lax pair.

Analogy with rotating tops. If the CH(2,2) problem specified here for a right-invariant Lagrangian on $\mathfrak{X} \ltimes \mathfrak{X}$ had been expressed instead on $\mathfrak{so}(3) \ltimes \mathfrak{so}(3)$ for a *left-invariant* Lagrangian, the result would have been interpreted as the dynamics of a rotating top in a potential force field, as discussed in [3]. The dynamics in this case is expressible in Hamiltonian form as

$$\begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 \times & \mathbf{q}_2 \times \\ \mathbf{q}_2 \times & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \mathbf{q}_1 = \mathbf{u}_0 \\ \delta h / \delta \mathbf{q}_2 = \mathbf{u}_1 \end{bmatrix}, \quad (3.20)$$

for angular momenta $(\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ and their corresponding angular velocities $(\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{R}^3 \times \mathbb{R}^3$ and Hamiltonian $h = \frac{1}{2}(\mathbf{q}_1 \cdot \mathbf{u}_0 + \mathbf{q}_2 \cdot \mathbf{u}_1)$. This Hamiltonian matrix defines a Lie-Poisson bracket on the dual of the semidirect-product Lie algebra $\mathbb{R}^3 \ltimes \mathbb{R}_0^3$ in which the first \mathbb{R}^3 acts on itself and on the second, \mathbb{R}_0^3 , by vector cross product, while the action of the second \mathbb{R}_0^3 on itself is by simple vector addition.

4 Other examples of two-component CH generalizations

In this section we discuss two other examples of equations in the integrable CH2 hierarchy. These are: (i) the equation denoted CH(2,-1) in the CH2 hierarchy that occupies the “Dym position” in the KdV hierarchy; and (ii) the CH(2,1) equations with two time variables.

4.1 The CH2 Dym equation or CH(2, -1)

The CH(2,2) system (3.5) – (3.8) sits in a hierarchy of integrable equations that share the same spectral problem (3.3), as well as the main representative CH(2,1). There are other members of this hierarchy, for which $U(x, \lambda)$ contains *positive* powers of λ . A simple interesting example of this kind has

$$Q = -\lambda^2 \rho^2 + \lambda q + 1/4 \quad \text{and} \quad U = \lambda u, \quad (4.1)$$

Extending our notation, we can denote the resulting system as CH(2,-1).³ Substituting (4.1) into (2.5) leads to equations denoted as CH(2,-1):

$$\rho_t + \left(\frac{q}{\rho^2} \right)_x = 0, \quad (4.2)$$

$$q_t - \left((1 - \partial_x^2) \frac{1}{\rho} \right)_x = 0. \quad (4.3)$$

where we have used $u\rho = K$, obtained from the differential relation arising in the λ^3 term, and have set the constant value $K = -2$. Some of its solution behavior is shown in Figure 4.1

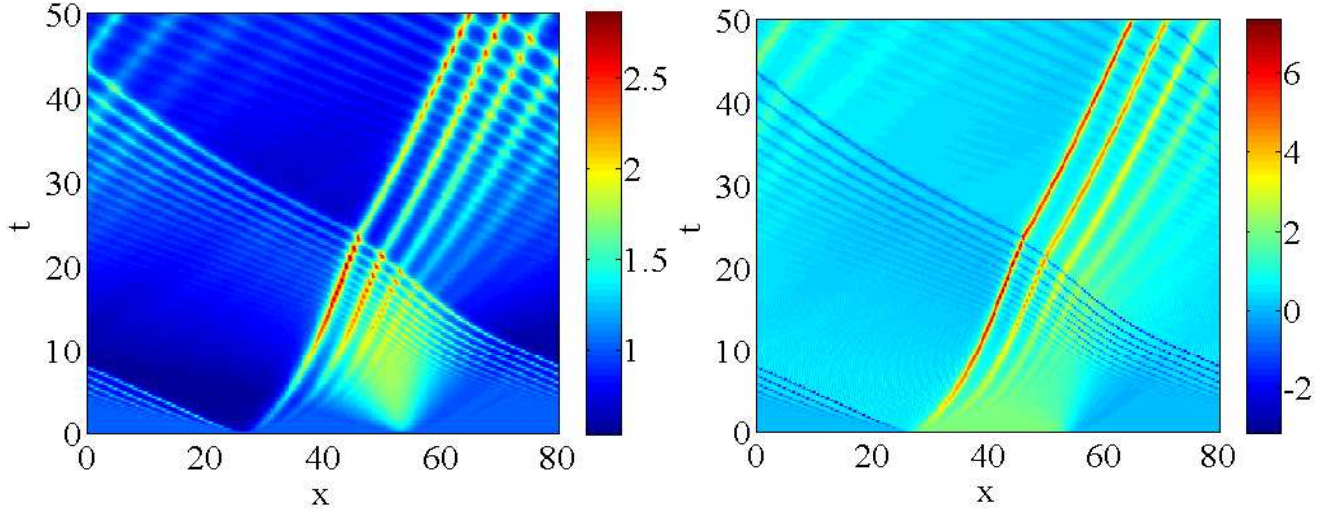


Figure 4.1: Results are shown for the evolution of the CH(2,-1) system (4.2-4.3) with $\epsilon = -1$ for ρ (left) and q (right), arising in a periodic domain of length $L = 80$ from initial conditions that represent a dam-break $q(x, 0) = \tanh((x - L_1)/\alpha) - \tanh((x - L_2)/\alpha)$, $\rho(x, 0) = 1$ with $\alpha = 1$, $L_1 = L/3$, $L_2 = 2L/3$. Soliton solutions are seen to emerge and propagate in both directions. The head-on collision process produces a slight refraction of the soliton trajectories, unlike the CH(2,1) case. Figures are courtesy of V. Putkaradze.

Remarks.

Here are a few remarks about the CH(2,-1) system in equations (4.2) and (4.3):

- This coupled nonlinear system is at the position in the CH2 hierarchy that corresponds to the modified Dym equation, first introduced as a tri-Hamiltonian system in [50].
- The CH(2,-1) equations combine to produce the **nonlinear wave equation**

$$q_{tt} = (1 - \partial_x^2) \left(\partial_x \frac{1}{\rho^2} \partial_x \frac{1}{\rho^2} \right) q. \quad (4.4)$$

Linearizing this equation around $q = 0$ and $\rho = 1$ yields the dispersion relation for a plane wave $\exp(i(kx - \omega t))$ with wave number k and frequency ω as

$$\omega^2(k) = (1 + k^2)k^2.$$

Accordingly, the phase speed of the linearized plane waves is $\omega/k = \sqrt{1 + k^2}$, so the higher wave numbers travel faster. This type of dispersion relation is not unfamiliar: it is the same as for time-dependent Euler-Bernoulli theory for an elastic beam with both bending and vibration response [52].

³ It is possible to extend the expansions in powers of λ in (2.3), (2.4) in both directions for both Q and U . The equation chosen for analysis here is only a single step in this extension.

- The CH(2,-1) **travelling wave solutions** $v(\xi) = 1/\rho(\xi)$ with $\xi = x - ct$ for $c > 0$ conserve the energy E given by

$$2E = (v')^2 - \left(\frac{c}{v} - I\right)^2 - (v + J)^2, \quad (4.5)$$

with integration constants I and J defined by

$$I = \frac{c}{v} - qv^2 \quad \text{and} \quad J = v'' - v + cq. \quad (4.6)$$

For the travelling wave $E = 0$, and when $I = 0 = J$, as well, then the solution is given by

$$c\rho(\xi) = \sqrt{\text{sech}(2(\xi - \xi_0))}, \quad \xi_0 = \text{constant and} \quad q(\xi) = c\rho^3(\xi). \quad (4.7)$$

This is a confined travelling wave pulse in both ρ and q .

- The coupled CH(2,-1) system (4.2-4.3) may also be written in **Hamiltonian form**, as

$$\partial_t \begin{bmatrix} \rho \\ \epsilon q \end{bmatrix} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x - \partial_x^3 \end{bmatrix} \begin{bmatrix} -q/\rho^2 = \delta h / \delta \rho \\ 1/\rho = \delta h / \delta q \end{bmatrix}, \quad \text{with} \quad h := \int (q/\rho) dx. \quad (4.8)$$

This Hamiltonian operator yields the Poisson bracket dynamics,

$$\frac{dF}{dt} = \{F, H\} = \int \left(\frac{\delta F}{\delta \rho} \partial_x \frac{\delta H}{\delta \rho} + \frac{\delta F}{\delta q} (\partial_x - \partial_x^3) \frac{\delta H}{\delta q} \right) dx. \quad (4.9)$$

The energy conservation law may be expressed in conservative form as

$$\partial_t \left(\frac{q}{\rho} \right) + \partial_x \left(-\frac{1}{2} \left(\frac{q}{\rho^2} \right)^2 + \frac{1}{2\rho^2} - \frac{1}{\rho} \partial_x^2 \frac{1}{\rho} + \frac{1}{2} \left(\partial_x \frac{1}{\rho} \right)^2 \right) = 0. \quad (4.10)$$

- The CH(2,-1) equations (4.2) and (4.3) may also be **written in n dimensions** as

$$\rho_t + \nabla \cdot \left(\frac{\mathbf{q}}{\rho^2} \right) = 0, \quad (4.11)$$

$$\mathbf{q}_t - \nabla \cdot \left((1 - \nabla^2) \frac{1}{\rho} \right) = 0, \quad (4.12)$$

for $\rho \in \mathbb{R}$ and $\mathbf{q} \in \mathbb{R}^n$.

- The consistency among the CH(n,k) equations can be demonstrated by combining them. The choice, $Q = \epsilon_1 \lambda^2 \rho^2 + \lambda q + \frac{1}{4}$ and $U = -\frac{1}{2\lambda} + u + 2\epsilon_2 \lambda / \rho$ with $\epsilon_{1,2} = \pm 1$ in (2.5), for example, produces another nonlinear integrable system:⁴

$$\epsilon_1 \left(\rho_t + (u\rho)_x \right) + \epsilon_2 \left(\frac{q}{\rho^2} \right)_x = 0, \quad (4.13)$$

$$q_t + (\partial_x q + q \partial_x) u - \epsilon_1 \rho \rho_x + \epsilon_2 \left((1 - \partial_x^2) \frac{1}{\rho} \right)_x = 0, \quad (4.14)$$

with differential relation $q_x = u_x - u_{xxx}$. This system reduces to CH2 for $\epsilon_2 = 0$.

⁴ One may compare this with $Q = -\lambda^2 \rho^2 + \lambda q + 1/4$ and $U = \lambda u$ for CH(2,-1) in (4.1).

4.2 Equations in the CH2 hierarchy with two time variables

There is also an integrable CH2 system with two ‘time’ variables (t and y). In particular, consider the system⁵ The two-time CH2 system with $m = U_x - U_{xxx}$ is

$$m_t + 2U_{yx}m + (U_y + \gamma)m_x + \rho\rho_y = 0, \quad (4.15)$$

$$\rho_t + \left((U_y + \gamma)\rho \right)_x = 0. \quad (4.16)$$

This system can be written equivalently in a hydrodynamic form as

$$(m/\rho^2)_t + (U_y + \gamma)(m/\rho^2)_x = -\rho^{-1}\rho_y, \quad (4.17)$$

$$\rho_t + \left((U_y + \gamma)\rho \right)_x = 0, \quad (4.18)$$

which shows that it has only one characteristic velocity, $dx/dt = (U_y + \gamma)$.

The two-time CH2 system can also be written as the compatibility condition for the following linear system (**Lax pair**) with a constant spectral parameter ζ :

$$\Psi_{xx} = \left(-\zeta^2\rho^2 + \zeta m + \frac{1}{4} \right) \Psi, \quad (4.19)$$

$$\Psi_t - \frac{1}{2\zeta}\Psi_y = -(U_y + \gamma)\Psi_x + \frac{1}{2}U_{yx}\Psi. \quad (4.20)$$

The first equation in this system is the spectral problem (3.3) of the CH2 hierarchy. The second equation introduces the other ‘time’ derivative, with respect to y . The system (4.15),(4.16) appears on setting

$$\left(\partial_t - \frac{1}{2\zeta}\partial_y \right) \Psi_{xx} = \partial_x^2 \left(\Psi_t - \frac{1}{2\zeta}\Psi_y \right),$$

then using (4.19),(4.20) to eliminate higher derivatives and assuming $\zeta_t = 0 = \zeta_y$.

Remarks.

- Perhaps not unexpectedly, the corresponding modification of the linear system (2.1), (2.2) yields a two-time version of the entire CH(n,k) hierarchy in (2.8).
- The integrable system of two-time CH2 equations (4.15),(4.16) reduces to CH2 for $x = y$ and $u = U_x$.
- Likewise, the special case $\gamma = 0 = \omega$ with initial condition $\rho = 0$ admits N -peakon solutions,

$$m(x, t, y) = \sum_{a=1}^N p_a(t, y) \delta(x - q_a(t, y)), \quad (4.21)$$

with two ‘time’ variables (t and y).

- The amplitudes of m and ρ as functions of x and y in Figure 4.3 show modulations along the crest of the two-dimensional solitons as they form under the two-time CH2 dynamics. .

⁵A two-time version of the CH equation has been considered previously in [37].

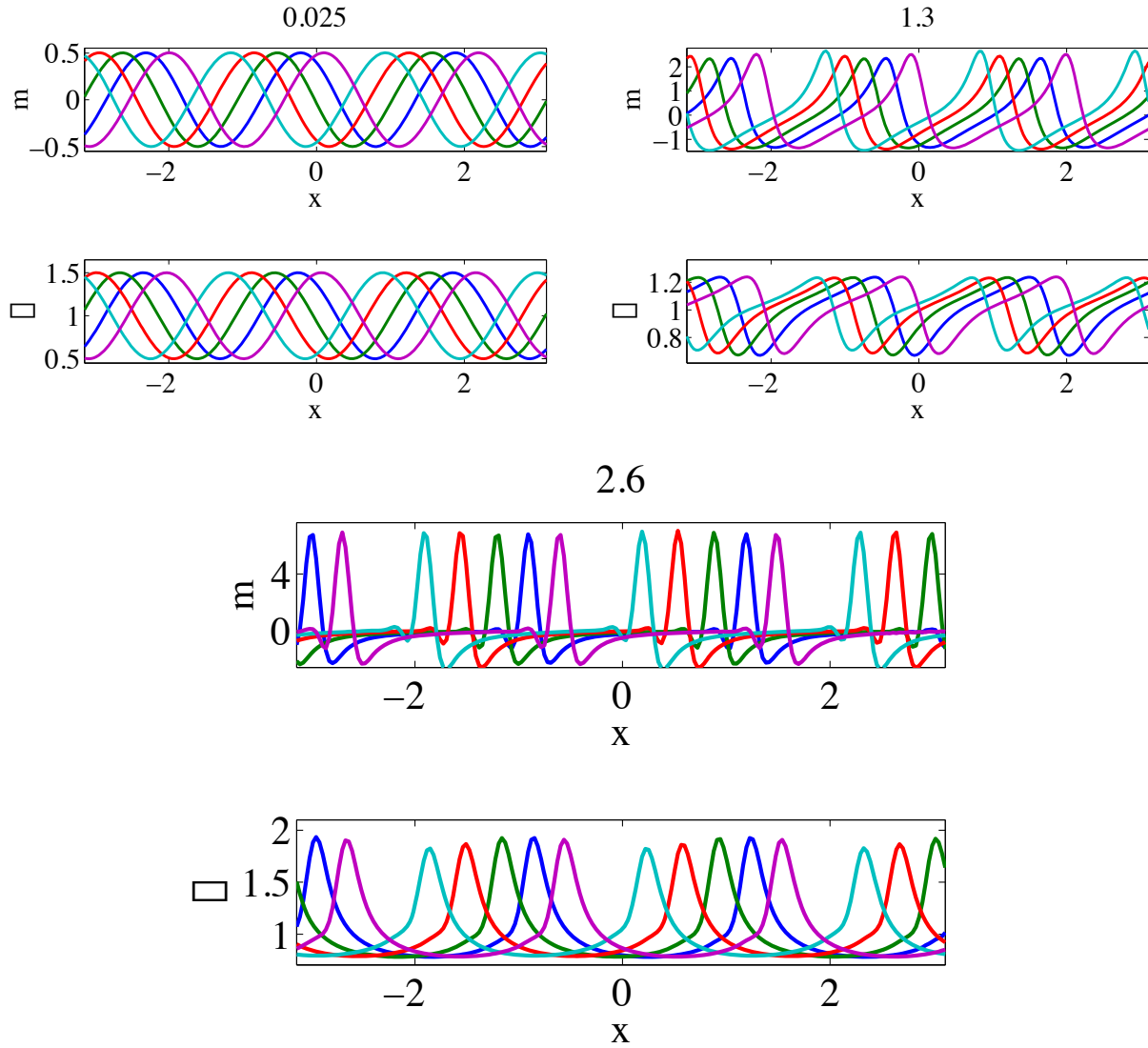


Figure 4.2: An initially sinusoidal wave train (plotted with m above and ρ below at five values of y shown in colours) concentrates into steeper, larger nonlinear waves under the dynamics of the two-time CH2 equations (4.15),(4.16). Figures are courtesy of L. Ó Náráigh.

Graded Lie algebra structure for the CH(n,k) chain with $n \geq 3$ and $k > 0$. A new feature of the CH(n,k) chain may be recognized for $n \geq 3$ and $k > 0$. In that case, the Hamiltonian operator reveals its character as the Lie-Poisson operator defined on the dual space of a *graded Lie algebra*. For example, in the case of CH(3,1) the system in the first line of (2.8) may be written in Lie-Poisson Hamiltonian form, as

$$\begin{bmatrix} \partial_t q_1 \\ \partial_t q_2 \\ \partial_t q_3 \end{bmatrix} = - \begin{bmatrix} \partial_x q_1 + q_1 \partial_x & \partial_x q_2 + q_2 \partial_x & \partial_x q_3 + q_3 \partial_x \\ \partial_x q_2 + q_2 \partial_x & \partial_x q_3 + q_3 \partial_x & 0 \\ \partial_x q_3 + q_3 \partial_x & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta q_1 = u_0 \\ \delta h / \delta q_2 = -\frac{1}{2} \\ \delta h / \delta q_3 = 0 \end{bmatrix}. \quad (4.22)$$

For $q_3 = -\rho^2$ this generalizes the CH(2,1) equation to three components. In hindsight, we see that we could have written the CH(2,1) equation in the same Lie-Poisson Hamiltonian form, as

$$\begin{bmatrix} \partial_t q_1 \\ \partial_t q_2 \end{bmatrix} = - \begin{bmatrix} \partial_x q_1 + q_1 \partial_x & \partial_x q_2 + q_2 \partial_x \\ \partial_x q_2 + q_2 \partial_x & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta q_1 = u_0 \\ \delta h / \delta q_2 = -\frac{1}{2} \end{bmatrix}, \quad (4.23)$$

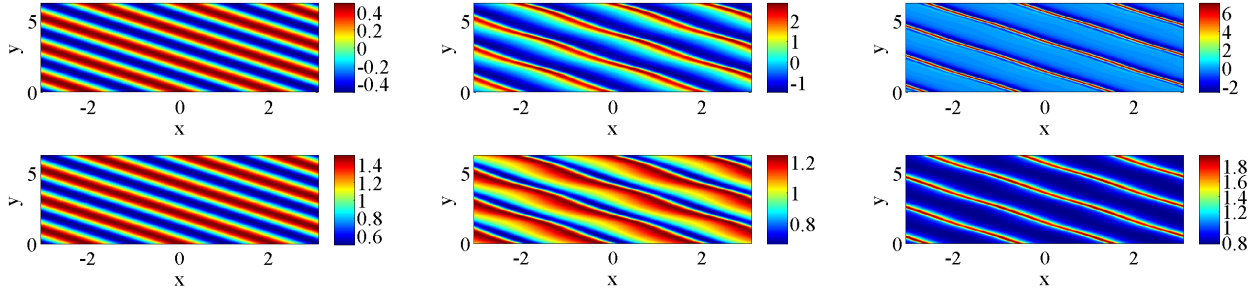


Figure 4.3: An initially sinusoidal wave train in the (x, y) plane (whose amplitudes are shown by colour bars) with m above and ρ below, steepens and modulates along the crests as it grows into a sequence of nonlinear wave packets under the dynamics of the two-time CH2 equations (4.15),(4.16). Figures are courtesy of L. Ó Náirigh.

for the Hamiltonian $h(q_1, q_2) = \frac{1}{2} \int q_1 (1 - \partial_x^2)^{-1} q_1 - q_2 dx$. Likewise, in the case of CH(3,2) the system becomes,

$$\begin{bmatrix} \partial_t q_1 \\ \partial_t q_2 \\ \partial_t q_3 \end{bmatrix} = - \begin{bmatrix} \partial_x q_1 + q_1 \partial_x & \partial_x q_2 + q_2 \partial_x & \partial_x q_3 + q_3 \partial_x \\ \partial_x q_2 + q_2 \partial_x & \partial_x q_3 + q_3 \partial_x & 0 \\ \partial_x q_3 + q_3 \partial_x & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta q_1 = u_0 \\ \delta h / \delta q_2 = u_1 \\ \delta h / \delta q_3 = -\frac{1}{2} \end{bmatrix}. \quad (4.24)$$

This grading of the Hamiltonian operator according to the weight n reveals the new feature. For CH(3,k) with $k > 0$, the Hamiltonian operator in (4.24) defines a Poisson bracket on the dual of the Lie algebra of three-component vector fields $(X_1, X_2, X_3) \in \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3$ defined by their graded commutation relation

$$[(X_1, X_2, X_3), (Y_1, Y_2, Y_3)] = ([X_1, Y_1], [X_1, Y_2] + [X_2, Y_1], [X_1, Y_3] + [X_2, Y_2] + [X_3, Y_1]). \quad (4.25)$$

Dual coordinates are $q_1 \in \mathfrak{X}_1^*(\mathbb{R})$, $q_2 \in \mathfrak{X}_2^*(\mathbb{R})$ and $q_3 \in \mathfrak{X}_3^*(\mathbb{R})$. The Lie-Poisson bracket above may now be written as

$$\{F, H\} = - \left\langle (q_1, q_2, q_3), \left[\left(\frac{\delta F}{\delta q_1}, \frac{\delta F}{\delta q_2}, \frac{\delta F}{\delta q_3} \right), \left(\frac{\delta H}{\delta q_1}, \frac{\delta H}{\delta q_2}, \frac{\delta H}{\delta q_3} \right) \right] \right\rangle, \quad (4.26)$$

in terms of the graded commutation relation (4.25) and the L^2 pairing $\langle \cdot, \cdot \rangle$ between the graded Lie algebra and its dual.

Remarks.

- A Lie-Poisson bracket defined on the dual of the same graded Lie algebra also appears in plasma theory, at third order in the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations [46].
- From the graded Lie-algebra action (4.25) and the Lax pair in (2.1) - (2.2), it is clear how to extend the pattern to higher order and thus include the more deeply nested systems in the CH(n,k) chain.
- The semidirect product action in (3.10) is also an instance of the graded Lie-algebra action.
- This type of weighted Lie-Poisson bracket is also encountered in the Hamiltonian formulation of dynamics in the BBGKY hierarchy for ideal plasma physics [46].
- A graded extension of (3.20) analogous to that for the Hamiltonian operator in (4.24) is also available for coupling with additional $so(3)^*$ angular momenta.
- The properties of the full Hamiltonian structure for the CH(n,k) chain in (2.8) consisting of n evolution equations with $|k|$ differential relations that allow $k < 0$ will be discussed elsewhere. (For $k < 0$, the grading runs in the ‘opposite direction’ in a certain sense.)

5 Conclusion

Main results of the paper. We first formulated the Lax pair consisting of the energy-dependent isospectral problem and evolution equation (2.1), (2.2), whose compatibility yields the integrable family of CH(n,k) systems (2.15)

with n components (momenta) and $1 \leq |k| \leq n$ velocities. After looking at several examples among the CH(n,k) multi-component equations, we investigated some of the other equations of the CH2 hierarchy and found geometrical comparisons with systems of coupled spinning tops, as well as fluids because of the semidirect-product nature of their Lie-Poisson brackets. In particular, the integrable CH(2,2) equations in (3.9) with Hamiltonian matrix in equation (3.19) were seen to be analogous to the finite-dimensional equations for a spinning top in a potential force field [3, 23, 24], whose Lie-Poisson bracket is dual to the semidirect-product Lie algebra $\mathfrak{so}(3) \ltimes \mathfrak{so}(3)_0$, in which the second entry is treated simply as a vector space, as discussed in §3.3.

Section §4 provided additional examples of other integrable equations in the CH2 hierarchy, such as the CH(2, -1) system and the two-time version of the CH(2,1) equations. The CH(n,k) systems arising from (2.1)-(2.3) with negative values of k were found to show quite different Hamiltonian structures from their corresponding systems with positive values of k .

Properties of the CH(n,k) systems. Several properties of the CH(n,k) systems were identified in the course of this work. These included the following.

- (i) The differential relations in the middle equation of (2.8) involved both of the compatible Poisson operators in the biHamiltonian structure (1.11). In both CH(2,1) and CH(2,2) the differential relations defined the momenta dual to the velocity vector fields.
- (ii) The Hamiltonian structure for the CH(n,k) chain with $k > 0$ in (2.8) consisting of n evolution equations and $k + 1$ differential relations possesses a *graded* Lie algebra structure, which became evident for $n \geq 3$. In hindsight, looking at (3.19), the semidirect-product Lie algebra structure for $n = 2$ in (4.23) could have already been understood as being graded. For $k < 0$ again both of the compatible Poisson operators appear.
- (iii) The sample numerical simulations shown for the CH(2,1), CH(2,2) and CH(2,-1) equations revealed challenging properties for future investigation, such as different types of collision behavior. In the case of CH(2,1) with two times, the simulations also revealed modulation of the waves along their crests during the formation of the soliton trains. These sample numerical simulations provided insight into the fascinating pulse-like solution behavior of the CH(n,k) equations and invited further investigation.

Future challenges. The CH(n,k) family of integrable partial differential equations (PDE) discussed here offer many interesting challenges for future research, particularly in determining and analyzing their solution behavior and possible physical applications. Besides CH(2,1) which may be interpreted as a shallow water system the CH(2,2) equations in §3, the CH(2,-1) system in §4.1 and the two-time CH2 equations in §4.2 all offer new challenges for physical interpretation and mathematical analysis. For example, one may expect the continuing interest in wave-breaking analysis for CH and CH2 to extend also to the other integrable PDE in the rest of the CH(n,k) family, including, e.g., the CH(2,-2) system, which was not discussed here. The numerical simulation of these integrable PDE, and the formulation and analysis of their discrete versions can also be expected to attract attention in future endeavors. Finally, the multi-dimensional extensions and deeper geometrical aspects of these new integrable PDE also pose interesting challenges for future research.

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References

- [1] Acheson, D. J. *Elementary Fluid Dynamics*. Oxford University Press (Oxford, 1990).
- [2] Antonowicz, M., Fordy, A. P. and Liu, Q. P. Energy-dependent third-order Lax operators. *Nonlinearity* **4** (1991) 669 – 684.
- [3] Bogoyavlensky, O. I. *Overtaking solitons: Nonlinear integrable equations*, “Nauka”, Moscow, 1991. 320 pp. ISBN: 5-02-014620-X.
- [4] Boutet de Monvel, A. and Shepelsky, D.: Riemann-Hilbert approach for the CH equation on the line, *C.R. Math. Acad. Sci. Paris* **343** (2006) 627–632.
- [5] Bruveris, M., Gay-Balmaz, F., Holm, D. D. and Ratiu, T. S. The momentum map representation of images. *J. Nonlin. Sci.* To appear 2010.
- [6] Camassa, R. and Holm, D. D. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664 (1993)
- [7] Camassa, R., Holm, D. and Hyman, J. A new integrable shallow water equation. *Adv. Appl. Mech.* **31**, (1994)
- [8] Chen M., Liu S.-Q. and Zhang Y. A two-component generalization of the Camassa-Holm equation and its solutions. *Lett. Math. Phys.* **75** (2006) 1–15; nlin.SI/0501028.
- [9] Chen, R.M. and Liu Y. Wave Breaking and Global Existence for a Generalized Two-Component Camassa-Holm System, *International Mathematics Research Notices*, Article ID rnq118, 36 pages. (2010) doi10.1093/imrn/rnq118
- [10] Clarkson, P.A., Gordo, P.R. and Pickering, A. Multicomponent equations associated to non-isospectral scattering problems. *Inverse Problems* **13** (1997) 1463–1476.
- [11] Constantin, A. On the scattering problem for the CH equation. *Proc. R. Soc. Lond. A* **457**, 953–970 (2001)
- [12] Constantin A., Gerdjikov V. and Ivanov R. Inverse scattering transform for the CH equation, *Inv. Problems* **22** (2006), 2197–2207; arXiv nlin/0603019v2 [nlin.SI].
- [13] Constantin, A. and Ivanov, R.I. Poisson structure and Action-Angle variables for the CH equation, *Lett. Math. Phys.* **76**, 93–108 (2006); nlin.SI/0602049
- [14] Constantin, A. and Ivanov, R.I. On an integrable two-component Camassa-Holm shallow water system. *Phys. Lett. A* **372** (2008), 7129–7132.
- [15] Dai, H.-H. Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod. *Acta Mech.* **127**, 193–207 (1998)
- [16] Dullin H. R., Gottwald G. A. & Holm D. D. CH, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves, *Fluid Dynam. Res.* **33** (2003), 73–95.
- [17] Dullin H. R., Gottwald G. A. & Holm D. D. On asymptotically equivalent shallow water wave equations, *Physica* **190D** (2004), 1–14.
- [18] Escher J., Lechtenfeld O. and Yin, Z. Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation. *Discrete Contin. Dyn. Syst.* **19** (2007) 493–513.
- [19] Estévez, P. G. and Prada, J. Hodograph transformations for a Camassa-Holm hierarchy in $2 + 1$ dimensions. *J. Phys. A* **38** (2005) 1287–1297.

- [20] Falqui, G. On a Camassa-Holm type equation with two dependent variables. *J. Phys. A* **39** (2006), 327–342.
- [21] Gesztesy, F. and Holden, H. Soliton Equations and Their Algebro-Geometric Solutions, Volume I: (1+1)-Dimensional Continuous Models, Cambridge studies in advanced mathematics, volume 79. Cambridge: Cambridge University Press, 2003
- [22] Gui G. and Liu Y. On the global existence and wave-breaking criteria for the two-component Camassa-Holm system. *J. Funct. Anal.* **258** (2010) 4251–4278.
- [23] Holm, D. D. *Geometric Mechanics: I Dynamics and Symmetry*, World Scientific Imperial College Press, Singapore, (2008), ISBN 978-1-84816-195-5.
- [24] Holm, D. D. *Geometric Mechanics: II Rotating, Translating and Rolling*, World Scientific Imperial College Press, Singapore, (2008), ISBN 978-1-84816-155-9.
- [25] Holm D.D. Peakons, in *Encyclopedia of Mathematical Physics*, eds. J.-P. Francoise, G.L. Naber and Tsou S.T. Oxford Elsevier, 2006 (ISBN 978-0-1251-2666-3), volume **4** pages 12–20.
- [26] Holm, D. D. and Ivanov, R. I. Smooth and peaked solitons of the CH equation, *J. Phys A Math. and Theor.*, Special issue on current trends in integrability and nonlinear phenomena, Expected online publication: October 2010.
- [27] Holm, D. D. and J. E. Marsden [2004], Momentum maps and measure-valued solutions (peakons, filaments and sheets) for the EPDiff equation, in *The Breadth of Symplectic and Poisson Geometry*, A Festschrift for Alan Weinstein, 203-235, *Progr. Math.*, **232**, J. E. Marsden and T. S. Ratiu, Editors, Birkhäuser Boston, Boston, MA, 2004.
- [28] Holm, D. D., Marsden, J. E. and Ratiu, T. S. The Euler-Poincaré equations and semidirect products with applications to continuum theories. *Advances in Mathematics*, **137**(1):1–81, 1998.
- [29] Holm D. D., Ó Náirigh, L. and Tronci, C. Singular solutions of a modified two-component CH equation. *Phys. Rev. E* (3) **79** (2009), no. 1, 016601, 13 pp.
- [30] Holm D. D., Schmah, T. and Stoica, C. *Geometric mechanics and symmetry. From finite to infinite dimensions*. With solutions to selected exercises by D. C. P. Ellis. Oxford Texts in Applied and Engineering Mathematics, **12**. Oxford University Press, Oxford, 2009.
- [31] Holm, D. D. and Tronci, C. Geodesic Vlasov equations and their integrable moment closures. *J. Geom. Mech.* **1** (2009) 181–208.
- [32] Holm, D.D. and Tronci, C. Geodesic flows on semidirect-product Lie groups: geometry of singular measure-valued solutions, *Proc. R. Soc. A* **465** (2009) 457476
- [33] Holm D. D., Trounev, A. and Younes, L. The Euler-Poincaré theory of metamorphosis. *Quart. Appl. Math.* **67** (2009) 661–685.
- [34] Henry, D. J. Infinite propagation speed for a two component Camassa-Holm equation. *Discrete Contin. Dyn. Syst. Ser. B* **12** (2009) 597–606.
- [35] Ivanov R. I. Extended CH hierarchy and conserved quantities, *Z. Naturforsch.*, **61a** (2006) pp. 133–138; nlin.SI/0601066.
- [36] Ivanov, R. I. Two-component integrable systems modelling shallow water waves: the constant vorticity case. *Wave Motion* **46** (2009), 389–396.
- [37] Ivanov, R. I. Equations of the Camassa-Holm hierarchy, *Theor. and Math. Phys.*, **160** (2009) 952–959.

- [38] Jaulent M. and Jean C. The inverse s -wave scattering problem for a class of potentials depending on energy. *Comm. Math. Phys.* **28** (1972) 177–220.
- [39] Johnson, R. S. CH, Korteweg-de Vries and related models for water waves. *J. Fluid. Mech.* **457**, 63–82 (2002)
- [40] Johnson, R. S. The CH equation for water waves moving over a shear flow. *Fluid Dynamics Research* **33**, 97–111 (2003)
- [41] Kaup, D. J. A higher-order water wave equation and the method for solving it, *Progr. Theor. Phys.* **54** (1975) 396–408.
- [42] Kraenkel, R. A., Senthilvelan, M. and Zenchuk, A. I. Lie symmetry analysis and reductions of a two-dimensional integrable generalization of the Camassa-Holm equation. *Phys. Lett. A* **273** (2000) 183–193.
- [43] Kuz'min, P. A. On two-component generalizations of the Camassa-Holm equation. *Mat. Zametki* **81** (2007) 149–152(Russian); translation in *Math. Notes* **81** (2007) 130–134.
- [44] Liu S.-Q. and Zhang Y. Deformations of semisimple bi-Hamiltonian structures of hydrodynamic type, *J. Geom. Phys.* **54** (2005) 427–53.
- [45] Magri, F. A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** (1978) 1156–1262.
- [46] J. E. Marsden, P. J. Morrison, and A. Weinstein, The Hamiltonian structure of the BBGKY hierarchy equations. In “Fluids and plasmas: geometry and dynamics” *Contemp. Math.* **28** (1984), Amer. Math. Soc., pp. 115–124.
- [47] Martínez Alonso, L. and Shabat, A. B. Energy-dependent potentials revisited: a universal hierarchy of hydrodynamic type. *Phys. Lett. A* **300** (2002) 58–64.
- [48] Martínez Alonso, L. and Shabat, A. B. Towards a theory of differential relations of a hydrodynamic hierarchy. *J. Nonlinear Math. Phys.* **10** (2003) 229–242.
- [49] Misiolek, G. A shallow water equation as a geodesic flow on the Bott-Virasoro group. *J. Geom. Phys.* **24** (1998) 203–208.
- [50] Olver, P. and Rosenau, P. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E* (3) **53** (1996) 1900–1906.
- [51] Qiao, Z. The Camassa-Holm Hierarchy, N -Dimensional Integrable Systems, and Algebro-Geometric Solution on a Symplectic Submanifold, *Commun. Math. Phys.* **239** 309–341.
- [52] Reddy, J.N. Nonlocal theories for bending, buckling and vibration of beams. *Internat. J. of Eng. Sci.* **45** (2007) 288–307.
- [53] Sattinger, D. H. and Szmigielski, J. A Riemann-Hilbert problem for an energy dependent Schrödinger operator. *Inverse Problems* **12** (1996) 1003–1025.
- [54] Shabat A. and Martínez Alonso L. On the prolongation of a hierarchy of hydrodynamic chains. In *New trends in integrability and partial solvability* (ed. A.B. Shabat et al.), Proceedings of the NATO advanced research workshop, Cadiz, Spain 2002, NATO Science Series, Kluwer Academic Publishers, Dordrecht: 2004, pp. 263–280.
- [55] C. Tronci, Geometric dynamics of Vlasov kinetic theory and its moments, arXiv:0804.3676v1, <http://arxiv.org/pdf/0804.3676>
- [56] Zhang, P. and Liu, Y. Stability of Solitary Waves and Wave-Breaking Phenomena for the Two-Component CH System, *International Mathematics Research Notices* **2010** (2010), No. 11, pp. 1981–2021.